

GRAPHICAL STRUCTURE OF CONDITIONAL INDEPENDENCIES IN DETERMINANTAL POINT PROCESSES

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ABSTRACT. Determinantal point process have recently been used as models in machine learning and this has raised questions regarding the characterizations of conditional independence. In this paper we investigate characterizations of conditional independence. We describe some conditional independencies through the conditions on the kernel of a determinantal point process, and show many can be obtained using the graph induced by a kernel of the L -ensemble.

In recent years there have been several machine learning papers about the applications of determinantal point processes (DPP's) [4], [7], [8], [9]... An overview of theory, recent applications and problems in learning DPP's is given in a recent extensive survey [6] by Kulesza and Taskar.

In a private communication with Ben Taskar, one of the questions from survey [6] (see §7.3), that remains for future research, was brought to author's attention:

- *Is there a simple characterization of the conditional independence relations encoded by a DPP?*

This question arises naturally having in mind conditional independence structure models (see [12]), such as graphical models (see [11]) that are often used.

It turns out that, from the mathematical view point, elegant characterizations, similar to those in graphical models, exist. This paper provides two (main) characterizations:

- the block in a Schur complement of the kernel has to be a 0-block (Theorem 16, Proposition 17);
- we can use the structure of the graph induced by the kernel of the L -ensemble to read many conditional independencies in the process (Theorem 28, Proposition 30).

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1. INTRODUCTION TO THE MODEL

In this paper K will be a positive semi-definite $N \times N$ matrix. Let $\mathbf{0} \preceq K \preceq \mathbf{I}$, $\mathcal{Y} = \{1, \dots, N\}$. We call a random subset \mathbf{Y} of \mathcal{Y} a **determinantal point process** if the following holds

$$\Pr(A \subset \mathbf{Y}) = \det(K_A),$$

and by definition $\Pr(\emptyset \subset \mathbf{Y}) = 1$. (Where $K_A = [K_{ij}]_{i,j \in A}$.)

Basically, we have a set of N points, and we pick a random subset \mathbf{Y} of them. We model the probability that all the points in the subset A were chosen by $\det(K_A)$.

Instead of modeling with the **kernel** K , in practice a determinantal point process is modeled as an L -ensemble. The process \mathbf{Y} is called the L -ensemble with the **kernel** L if

$$\Pr(\mathbf{Y} = A) = \frac{\det(L_A)}{\det(L + I)},$$

where L is a positive semi-definite matrix.

Theorem 1. *An L -ensemble with kernel L is a DPP with the kernel*

$$K = L(L + I)^{-1} = I - (L + I)^{-1}.$$

Corollary 2. *For $\mathbf{0} \prec K \prec \mathbf{I}$, a DPP with a kernel K is an L -ensemble where*

$$(1) \quad L = K(I - K)^{-1} = (I - K)^{-1} - I.$$

The following proposition summarizes some useful results about DPP's (they are all proven in [6]). Through this text $K_{AB} = [K_{ij} : i \in A, j \in B]$.

Proposition 3. *Let \mathbf{Y} be a DPP over \mathcal{Y} with kernel K and $A \subset \mathcal{Y}$.*

- (a) *The process $\mathbf{Y}_A = \mathbf{Y} \cap A$ is a DPP with kernel K_A .*
- (b) *We have*

$$\Pr(A \subset \mathbf{Y}, B \cap \mathbf{Y} = \emptyset) = (-1)^{|B|} \det \begin{bmatrix} K_A & K_{AB} \\ K_{AB}^T & K_B - I \end{bmatrix}.$$

- (c) *The process $\mathcal{Y} \setminus \mathbf{Y}$ is a DPP with the kernel $I - K$.*

For more on results and properties of DPP's see [1] or §4 in [3].

In further text, we will assume $\mathbf{0} \prec K \prec \mathbf{I}$ and $\mathbf{0} \prec L$.

2. INDEPENDENCIES

Under which conditions for three disjoint subsets A, B, C of \mathcal{Y} we have¹

$$(2) \quad (A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y}) \mid (C \subset \mathbf{Y}).$$

¹We use the notation $S_1 \perp S_2 \mid S_3$ to denote that S_1 is independent of S_2 given S_3 .

This was investigated by Kulesza in [5], where the answer is given for the case $|A| = |B| = 1$. We will give a very general answer in Proposition 17.

2.1. Independence in DPPs. We will start with the case $C = \emptyset$. When is

$$(3) \quad (A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y})?$$

The following are some known technical results from matrix analysis (see [2]).

Lemma 4. *Let*

$$(4) \quad M_+ = \begin{bmatrix} U & V \\ V^T & W \end{bmatrix} \quad \text{and} \quad M_- = \begin{bmatrix} U & -V \\ -V^T & W \end{bmatrix}$$

be quadratic matrices.

(a) *If M_+ and M_- are symmetric matrices their eigenvalues are the same with the same multiplicity. Further their determinants are also the same.*

(b) *M_+ is positive definite if and only if M_- is positive definite.*

(c) *M_+ is positive definite, if and only if*

$$(5) \quad U - VW^{-1}V^T \quad \text{and} \quad W$$

are positive definite.

(d) *If W is non-singular, then*

$$\det(M_+) = \det(M_-) = \det(W) \det(U - VW^{-1}V^T)$$

Corollary 5. *If M_+ is a positive (semi)definite matrix so is*

$$M_0 = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0}^T & W \end{bmatrix}.$$

Proof. Follows from the fact that $M_0 = \frac{1}{2}(M_+ + M_-)$. □

The following technical lemma will be the key for conditional independencies.

Lemma 6. *Let A be a positive definite and B a positive semi-definite $N \times N$ matrices. If $\det(A + B) = \det A$, then $B = 0$.*

Proof. Since A is positive definite, there exists a positive definite matrix \sqrt{A} , such that $A = (\sqrt{A})^2$. Therefore, since $\det A = (\det \sqrt{A})^2$, we have

$$(6) \quad \det(I + A^{-1/2}BA^{-1/2}) = 1.$$

It is not hard to see that $A^{-1/2}BA^{-1/2}$ is a positive semi-definite matrix. Hence (6) is equivalent (using the eigenvalue decomposition)

$$(1 + \lambda_1) \dots (1 + \lambda_N) = 1,$$

where $\lambda_1, \dots, \lambda_N$ are eigenvalues of $A^{-1/2}BA^{-1/2}$. Since this matrix is positive semi-definite, $\lambda_j \geq 0$ for $j = 1, \dots, N$ and therefore we have $\lambda_1 = \dots = \lambda_N = 0$. Hence, $A^{-1/2}BA^{-1/2} = \mathbf{0}$ and the claim follows. □

Corollary 7. *Let*

$$M = \begin{bmatrix} U & V \\ V^T & W \end{bmatrix}.$$

If one of the following conditions holds

- (a) *M is positive definite;*
- (b) *U is positive definite and W is negative definite;*
- (c) *M is negative definite;*
- (d) *U is negative definite and W is positive definite;*

then the equality

$$\det \begin{bmatrix} U & V \\ V^T & W \end{bmatrix} = \det U \det W.$$

holds if and only if $V = \mathbf{0}$.

Proof. If $V = \mathbf{0}$ the claim is clear.

We will prove cases (a) and (b), cases (c) and (d) follow from them.

Assume $\det M = \det U \det W$. Using Lemma 4(d) we get $\det(M) = \det(W) \det(U - VW^{-1}V^T)$, and from the assumption we have

$$\det U = \det(U - VW^{-1}V^T).$$

(a): $A = U - VW^{-1}V^T$ is positive definite (by (5)) and $B = VW^{-1}V^T$ is positive semi-definite. By Lemma 6 we have $B = \mathbf{0}$. Now, let $V^T = [v_1, \dots, v_m]$. Since, $B = \mathbf{0}$, $v_j^T W^{-1} v_j = 0$, and since W^{-1} is positive definite we have $v_j = \mathbf{0}$ for $j = 1 \dots m$.

(b): Set $A = U$ and $B = V(-W^{-1})V^T$. Since $-W^{-1}$ is positive definite, B is positive semi-definite and by Lemma 6, $B = \mathbf{0}$. Using the same approach as in case (a) we get $V = \mathbf{0}$. \square

Theorem 8. *If K is a kernel for the determinantal point process \mathbf{Y} over \mathcal{Y} , A and B disjoint subsets of \mathcal{Y} , then $(A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y})$ if and only if $K_{AB} = \mathbf{0}$.*

Proof. By definition, we have $(A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y})$ if and only if

$$\Pr(A \cup B \subset \mathbf{Y}) = \Pr((A \subset \mathbf{Y}) \cap (B \subset \mathbf{Y})) = \Pr(A \subset \mathbf{Y}) \Pr(B \subset \mathbf{Y}).$$

This is equivalent to

$$\det K_{A \cup B} = \det \begin{bmatrix} K_A & K_{AB} \\ K_{AB}^T & K_B \end{bmatrix} = \det K_A \det K_B.$$

By Corollary 7, this holds if and only if $K_{AB} = \mathbf{0}$. \square

Corollary 9. *If K is a kernel for the determinantal point process \mathbf{Y} over \mathcal{Y} , A and B disjoint subsets of \mathcal{Y} , then $(A \cap \mathbf{Y} = \emptyset) \perp (B \cap \mathbf{Y} = \emptyset)$ if and only if $K_{AB} = \mathbf{0}$.*

Proof. By Proposition 3(c) $\mathcal{Y} \setminus \mathbf{Y}$ is DPP with kernel $I - K$. The claim now follows from Theorem 8. \square

Theorem 10. *If K is a kernel for the determinantal point process \mathbf{Y} over \mathcal{Y} , A and B disjoint subsets of \mathcal{Y} , then $(A \subset \mathbf{Y}) \perp (B \cap \mathbf{Y} = \emptyset)$ if and only if $K_{AB} = \mathbf{0}$.*

Proof. By Proposition 3 (b) we know that $(A \subset \mathbf{Y}) \perp (B \cap \mathbf{Y} = \emptyset)$ if and only if

$$\begin{aligned} \Pr(A \subset \mathbf{Y}, B \cap \mathbf{Y} = \emptyset) &= (-1)^{|B|} \det \begin{bmatrix} K_A & K_{AB} \\ K_{AB}^T & K_B - I \end{bmatrix} \\ &= \Pr(A \subset \mathbf{Y}) \Pr(B \cap \mathbf{Y} = \emptyset) = \det K_A (-1)^{|B|} \det(K_B - I). \end{aligned}$$

By Corollary 7 this is true if and only if $K_{AB} = \mathbf{0}$ \square

Using the same techniques as in last proofs, we can prove much more.

Theorem 11. *If K is a kernel for the determinantal point process \mathbf{Y} over \mathcal{Y} , A and B disjoint subsets of \mathcal{Y} , then the processes $\mathbf{Y}_A = \mathbf{Y} \cap A$ and $\mathbf{Y}_B = \mathbf{Y} \cap B$ are independent if and only if $K_{AB} = \mathbf{0}$.*

Proof. If \mathbf{Y}_A and \mathbf{Y}_B are independent, then $(A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y})$, and hence by Theorem 8 the claim follows.

Let $L^{A \cup B}$ denote the kernel of the L -ensemble of the process $\mathbf{Y} \cap (A \cup B)$. If $K_{AB} = \mathbf{0}$ we know that for $A_1 \subset A$ and $B_1 \subset B$ we have

$$\begin{aligned} \Pr(A \cap \mathbf{Y} = A_1, B \cap \mathbf{Y} = B_1) &= \det(L_{A_1 \cup B_1}^{A \cup B}) = \\ &= \det L_{A_1}^{A \cup B} \det L_{B_1}^{A \cup B} = \Pr(A \cap \mathbf{Y} = A_1) \Pr(B \cap \mathbf{Y} = B_1), \end{aligned}$$

since

$$L_{A_1 \cup B_1}^{A \cup B} = (I - K_{A_1 \cup B_1})^{-1} - I = \begin{bmatrix} (I - K_{A_1})^{-1} - I & 0 \\ 0 & (I - K_{B_1})^{-1} - I \end{bmatrix}. \quad \square$$

The following proposition summarizes the all the results from this subsection.

Proposition 12. *For a DPP with the kernel $\mathbf{0} \prec K \prec \mathbf{I}$, and A and B disjoint subsets of \mathcal{Y} the following statements are equivalent:*

- (a) $(A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y})$;
- (b) $(A \subset \mathbf{Y}) \perp (B \cap \mathbf{Y} = \emptyset)$;
- (c) $(A \cap \mathbf{Y} = \emptyset) \perp (B \cap \mathbf{Y} = \emptyset)$;
- (d) $\mathbf{Y}_A \perp \mathbf{Y}_B$;
- (e) $K_{AB} = \mathbf{0}$.

Remark. One might be tempted to think that if

$$(7) \quad (A_1 \subset \mathbf{Y}, A_2 \cap \mathbf{Y} = \emptyset) \perp (B_1 \subset \mathbf{Y}, B_2 \cap \mathbf{Y} = \emptyset)$$

then $K_{A_1 \cup A_2, B_1 \cup B_2} = \mathbf{0}$. However, this doesn't have to be true. Take

$$K = \begin{bmatrix} 0.05 & 0 & 0.1 \\ 0 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix}.$$

It is not hard to check that $\mathbf{0} \prec K \prec \mathbf{I}$. Set $A_1 = \{1\}$, $A_2 = \{2\}$, $C_1 = \{3\}$ and $C_2 = \emptyset$. Clearly, $K_{A_1 \cup A_2, C_1 \cup C_2} \neq \mathbf{0}$. However, by Proposition 3 (b)

$$\Pr(A_1 \subset \mathbf{Y}, A_2 \cap \mathbf{Y} = \emptyset, B_1 \subset \mathbf{Y}) = -\det \begin{bmatrix} 0.05 & 0 & 0.1 \\ 0 & -0.2 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} = 0.006,$$

is a product of $\Pr(A_1 \subset \mathbf{Y}, A_2 \cap \mathbf{Y} = \emptyset) = -\det \begin{bmatrix} 0.05 & 0 \\ 0 & -0.2 \end{bmatrix} = 0.01$ and $\Pr(B_1 \subset \mathbf{Y}) = 0.6$. Hence, in this case (7) is true.

2.2. Conditional independencies in DPP's. It is known that conditioned on the event $(A \subset \mathbf{Y}, B \cap \mathbf{Y} = \emptyset)$ the process \mathbf{Y} is a DPP. (See [6] or [1].)

Definition 13. If M is a square matrix and M_C is non-singular then we can define (the Schur complement of M)

$$(8) \quad M^C = M_{C^c} - M_{C^c, C} M_C^{-1} M_{C, C^c} = M_{C^c} - M_{C^c, C} M_C^{-1} M_{C^c, C}^T.$$

Remark. By Lemma 4(c) if K is positive definite, then K^C is positive definite. On the other hand, if $K \prec \mathbf{I}$, then, clearly, $\mathbf{I} - K^C = \mathbf{I} - K_{C^c} + K_{C^c, C} K_C K_{C, C^c} \succ \mathbf{0}$.

Lemma 14. For the determinantal point process \mathbf{Y} and some $C \subset \mathcal{Y}$ such that $|K_C| > 0$, for every $A \subset C^c$ we have

$$\Pr(A \subset \mathbf{Y} | C \subset \mathbf{Y}) = \det(K_A^C).$$

Hence $\mathbf{Y} \cap C^c$ given $(C \subset \mathbf{Y})$ is a DPP with the kernel K^C .

Proof. By definition,

$$\begin{aligned} \Pr(A \subset \mathbf{Y} | C \subset \mathbf{Y}) &= \frac{\Pr(A \subset \mathbf{Y}, C \subset \mathbf{Y})}{\Pr(C \subset \mathbf{Y})} = \frac{\Pr(A \cup C \subset \mathbf{Y})}{\Pr(C \subset \mathbf{Y})} \\ &= \frac{\det K_{A \cup C}}{\det K_C} = \frac{1}{\det K_C} \det \begin{bmatrix} K_A & K_{AC} \\ K_{AC}^T & K_C \end{bmatrix} \\ &\stackrel{\text{Lem. 4(d)}}{=} \det(K_A - K_{AC} K_C^{-1} K_{AC}^T) = \det(K_A^C). \end{aligned}$$

□

Theorem 15. For the determinantal point process \mathbf{Y} over \mathcal{Y} with the kernel K , and A, B, C disjoint subsets of \mathcal{Y} , then

$$(A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y}) | (C \subset \mathbf{Y})$$

is true if and only if $K_{AB}^C = 0$, i.e.

$$(9) \quad K_{AB} = \begin{cases} K_{AC} K_C^{-1} K_{BC}^T & C \neq \emptyset \\ 0 & C = \emptyset \end{cases}$$

Proof. If $C = \emptyset$ the claim follows from Theorem 8. When $C \neq \emptyset$ from Lemma 14 we know that $\mathbf{Y} \cap C^c | (C \subset \mathbf{Y})$ is a DPP with kernel K^C . Now, by Theorem 8 $(A \subset \mathbf{Y})$ and $(B \subset \mathbf{Y})$ are independent given $(C \subset \mathbf{Y})$ if and only if $K_{AB}^C = \mathbf{0}$. Since $K_{AB}^C = K_{AB} - K_{AC}K_C^{-1}K_{CB}$, the claim follows. \square

Using the same argumentation and Theorem 11 we have the following result.

Theorem 16. *If K is a kernel for the determinantal point process \mathbf{Y} over \mathcal{Y} , and A, B, C disjoint subsets of \mathcal{Y} , then $\mathbf{Y}_A = \mathbf{Y} \cap A$ and $\mathbf{Y}_B = \mathbf{Y} \cap B$ are independent given $(C \subset \mathbf{Y})$ if and only if $K_{AB}^C = 0$, i.e. (9) is true.*

The following is a generalization of the Proposition 12.

Proposition 17. *For a DPP with the kernel $\mathbf{0} \prec K \prec \mathbf{I}$, and A, B, C disjoint subsets of \mathcal{Y} the following statements are equivalent:*

- (a) $(A \subset \mathbf{Y}) \perp (B \subset \mathbf{Y}) | (C \subset \mathbf{Y})$;
- (b) $(A \subset \mathbf{Y}) \perp (B \cap \mathbf{Y} = \emptyset) | (C \subset \mathbf{Y})$;
- (c) $(A \cap \mathbf{Y} = \emptyset) \perp (B \cap \mathbf{Y} = \emptyset) | (C \subset \mathbf{Y})$;
- (d) $\mathbf{Y}_A \perp \mathbf{Y}_B | (C \subset \mathbf{Y})$;
- (e) $K_{AB}^C = 0$.

It is known (see for example (7.7.5) in [2]) that

$$(10) \quad (K^{-1})_{C^c} = (K^C)^{-1}.$$

Corollary 18. *Let \mathcal{Y} be a union of disjoint sets $\{i\}, \{j\}$ and $C = \mathcal{Y} \setminus \{i, j\}$. Then $K_{ij}^{-1} = 0$ if and only if $K_{ij}^C = 0$.*

Proof. Note that K^C is a 2×2 matrix. $K_{ij}^C = 0$ if and only if K^C is a diagonal matrix. This is so if and only if $(K^C)_{ij}^{-1} = 0 \stackrel{(10)}{=} (K^{-1})_{ij}$. \square

Corollary 19. *For $i, j \in \mathcal{Y}$ ($i \neq j$) \mathbf{Y}_i and \mathbf{Y}_j are independent given $\mathcal{Y} \setminus \{i, j\} \subset \mathbf{Y}$ if and only if*

$$K_{ij}^{-1} = 0.$$

Remark. Kulesza in [5] found that $i \in \mathbf{Y} \perp j \in \mathbf{Y} | (\mathcal{Y} \setminus \{i, j\} \subset \mathbf{Y})$ if and only if $K_{ij}^{-1} = 0$.

By Proposition 3 (c) $\mathcal{Y} \setminus \mathbf{Y}$ is a DPP with the kernel $I - K$. But the more interesting thing is that $\mathcal{Y} \setminus \mathbf{Y}$ is the L -ensemble with the kernel

$$(11) \quad \bar{L} = K^{-1} - I.$$

Now, the Corollary 19 can be restated in the terms of the matrix \bar{L} .

Corollary 20. *For $i, j \in \mathcal{Y}$ ($i \neq j$) \mathbf{Y}_i and \mathbf{Y}_j are independent given $\mathcal{Y} \setminus \{i, j\} \subset \mathbf{Y}$ if and only if*

$$\bar{L}_{ij} = 0.$$

Looking at the process $\mathbf{Y} = \mathcal{Y} \setminus (\mathcal{Y} \setminus \mathbf{Y})$ we have

Corollary 21. *For $i, j \in \mathcal{Y}$ ($i \neq j$) \mathbf{Y}_i and \mathbf{Y}_j are independent given $(\mathcal{Y} \setminus \{i, j\}) \cap \mathbf{Y} = \emptyset$ if and only if*

$$L_{ij} = 0.$$

3. COMPARISON TO GAUSSIAN GRAPHICAL MODELS

The way independence is encoded in matrices K and L is similar to way independence is encoded in covariance matrix Σ and precision matrix Σ^{-1} of the Gaussian random vector.

The question is, can we, from the structure of the matrix L , say more about conditional independencies in a DPP? Is there a similar result as in the Gaussian graphical models?

We will briefly review the results we have in Gaussian graphical models. We will assume $V = \{1, \dots, n\}$ and let the process

$$X = (X_v : v \in V)$$

be a normal random vector with expectation μ and a positive definite covariance matrix Σ .

Definition 22. *For a symmetric matrix M we will say that $G_M = (V, E_M)$ is a graph induced by the matrix M if the set of edges is given by*

$$E_M = \{\{i, j\} : M_{ij} \neq 0, i \neq j\}.$$

The following results are well known for Gaussian random vectors.

Theorem 23. (a) *For disjoint subsets A, B, C of V*

$$X_A \perp X_B | X_C$$

if and only if $\Sigma_{AB}^C = \mathbf{0}$.

(b) *For $k, j \in V$ with $k \neq j$*

$$X_k \perp X_j | X_{V \setminus \{k, j\}}$$

if and only if $\Sigma_{k, j}^{-1} = 0$.

Definition 24. (a) *We say that the process X has the pairwise Markov property with respect to the structure of the graph $G = (V, E)$ if $X_k \perp X_j | X_{V \setminus \{k, j\}}$ holds for all $\{k, j\} \notin E$.*

(b) *We say that the process X has the global Markov property if for A, B, C are disjoint subsets of V such that C separates A and B , i.e. any path starting at a vertex in A and ending in B has to go through a vertex in C , we have $X_A \perp X_B | X_C$.*

The following is a consequence of the famous Hammersley-Clifford Theorem and the fact that X has a positive density. (See §3.2.1. and Theorem 3.9. in [11].)

Theorem 25. *The process X has the pairwise Markov property with respect to graph $G = (V, E)$ if and only if it has the global Markov property with respect to G .*

Corollary 26. *X has the pairwise Markov property with respect to the structure of the graph $G_{\Sigma^{-1}} = (V, E_{\Sigma^{-1}})$. Further, X also has the global Markov property with respect to $G_{\Sigma^{-1}}$.*

Proof. From the definition, using Theorem 23. (b) the pairwise property follows. The global property follows from Theorem 25. \square

Theorem 27. *Let M be a positive definite $n \times n$ matrix, and $G_{M^{-1}} = (V, E_{M^{-1}})$ a graph induced by M^{-1} . If A, B, C are disjoint subsets of V such that C separates A and B , then*

$$M_{AB}^C = \mathbf{0}.$$

Proof. Let $Y \sim N(0, M)$. By Theorem 25, Y has the global Markov property with respect to the graph $G_{M^{-1}}$. Hence Y_A is independent of Y_B given Y_C , and by Theorem 23.(a) this is true if and only if $M_{AB}^C = \mathbf{0}$. \square

4. GRAPHS INDUCED BY THE L -ENSEMBLE

From the structure of the L -ensemble we can get some information about other conditional independencies. The following is a version of the global Markov property for L -ensembles.

Theorem 28. *Let the determinantal point process \mathbf{Y} be an L -ensemble and G_L be a graph induced by the kernel L . If A, B, C are disjoint subsets of V such that C separates A and B , then \mathbf{Y}_A is independent of \mathbf{Y}_B given that $\mathbf{Y} \cap C = \emptyset$.*

Proof. L has off-diagonal zeros in the same places as $(I - K)^{-1}$ (see (1)). By Theorem 27, we have that $(I - K)_{AB}^C = 0$. Hence, by Theorem 16, $(\mathcal{Y} \setminus \mathbf{Y}) \cap A$ and $(\mathcal{Y} \setminus \mathbf{Y}) \cap B$ are independent given $C \subset \mathcal{Y} \setminus \mathbf{Y}$. Hence, the claim follows. \square

Theorem 29. *Let the determinantal point process \mathbf{Y} be an L -ensemble and G_L be a graph induced by the kernel L . If A, B, C, D are disjoint subsets of V such that C separates A and B , then \mathbf{Y}_A is independent of \mathbf{Y}_B given that $\mathbf{Y} \cap C = \emptyset$ and $D \subset \mathbf{Y}$.*

Proof. Let D_A be all vertices from D that are connected to vertices from A with paths in G_L that do not pass through C . We set $D'_A = D \setminus D_A$. For $S_A \in \sigma(\mathbf{Y}_A)$ and $S_B \in \sigma(\mathbf{Y}_B)$, using the fact that C separates $A \cup D_A$ and

$B \cup D'_A$ and Theorem 28. we have

$$\begin{aligned}
& \Pr(S_A \cap S_B | \mathbf{Y} \cap C = \emptyset, D \subset \mathbf{Y}) \\
&= \Pr(S_A \cap S_B, D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset) / \Pr(D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset) \\
&= \frac{\Pr(S_A, D_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset) \Pr(S_B, D'_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)}{\Pr(D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)} \\
&= \frac{\Pr(S_A, D_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset) \Pr(D'_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)}{\Pr(D'_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset) \Pr(D'_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)} \\
&\quad \cdot \frac{\Pr(S_B, D'_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset) \Pr(D'_A \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)}{\Pr(D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)} \\
&= \frac{\Pr(S_A, D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)}{\Pr(D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)} \cdot \frac{\Pr(S_B, D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)}{\Pr(D \subset \mathbf{Y} | \mathbf{Y} \cap C = \emptyset)} \\
&= \Pr(S_A | \mathbf{Y} \cap C = \emptyset, D \subset \mathbf{Y}) \Pr(S_B | \mathbf{Y} \cap C = \emptyset, D \subset \mathbf{Y}).
\end{aligned}$$

□

Proposition 30. *Let the determinantal point process \mathbf{Y} be an L -ensemble and G_L be a graph induced by the kernel L . Let*

- A_1, \dots, A_n, C and D are disjoint subsets of \mathcal{Y} ;
- C separates sets A_1, \dots, A_n in G_L .

Then $\mathbf{Y}_{A_1}, \dots, \mathbf{Y}_{A_n}$ are independent given $C \cap \mathbf{Y} = \emptyset$ and $D \subset \mathbf{Y}$.

5. FINAL REMARKS

Proposition 17 gives necessary and sufficient conditions for conditional independencies, but it is not easy to practically check them. Further, estimating K is conjectured to be an NP-hard problem ([6]).

On the other hand, Theorem 28 gives us only sufficient conditions on the kernel L and given a sparse matrix L we can read many conditional independencies from its structure without any additional matrix transformations. Further, there are ways to estimate kernel L ([6]).

Although the independence induced by the graph structure is not as strong as in the case of graphical models, it still provides important information about the process and is useful for better understanding of this process.

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